

Effective-Mass Dirac Equation for Woods-Saxon Potential: Scattering, Bound States and Resonances

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Abstract

Approximate scattering and bound state solutions of the one-dimensional effective-mass Dirac equation with the Woods-Saxon potential are obtained in terms of the hypergeometric-type functions. Transmission and reflection coefficients are calculated by using behavior of the wave functions at infinity. The same analysis is done for the constant mass case. It is also pointed out that our results are in agreement with those obtained in literature. Meanwhile, an analytic expression is obtained for the transmission resonance and observed that the expressions for bound states and resonances are equal for the energy values $E = \pm m$.

Keywords: Scattering, Bound State, Resonance, Dirac Equation, Woods-Saxon Potential, Position-Dependent Mass

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I. INTRODUCTION

In order to get a complete information about quantum mechanical systems, one should study bound and scattering states in the presence of an external potential. Therefore, scattering problem has become interesting topic in relativistic/non-relativistic quantum mechanics. The problem has been studied within the framework of group theoretical approach [1] and investigated for the well-known potentials by applying different methods [2-18]. In the case of non-relativistic scattering problem, it has been showed that transmission and reflection coefficients take 1 and 0, respectively, as external potential has well-behaved at infinity for the zero energy limit [10-12]. However, reflection coefficient goes to zero while transmission coefficient goes to unity in the zero energy limit when external potential supports a half-bound state. This situation was called as transmission resonance by Bohm [13]. This phenomenon has been recently extended to the Dirac particle in Refs. [3, 14, 15]. In the zero-momentum limit, Dombey *et. al* displayed that the bound-state energy eigenvalue obtained for the Dirac particle in the presence of the Woods-Saxon potential is related to the transmission resonance appearing for a Dirac particle scattered by a potential well. Recently, Villalba *et. al* [6] have showed that relation between the bound-state energy eigenvalues and transmission resonances in view of the Woods-Saxon potential for the Klein-Gordon particle are the same as obtained for the Dirac particle [15].

On the other hand, solutions of the wave equations have, recently, become interesting in the view of position-dependent mass (PDM) formalism. Extensive applications of this formalism have been done in different areas of physics such as condensed matter physics and material science such as electronic properties of semiconductors [19], quantum dots [20], quantum liquids *etc.* [21-25]. In recent years, the scattering problem has been extended to the case where the mass depends on spatially coordinate [26-28]. Alhaidari has, recently, investigated solution of the Dirac equation in view of position-dependent mass for the Coulomb field [24]. In Ref. [27], the authors have studied the relativistic scattering in the Dirac equation by using the J-matrix method for the position-dependent mass. Panella *et. al* [28] have obtained a new exact solution of the effective-mass Dirac equation for the Woods-Saxon potential. The approximate solution of the Dirac equation with PDM for the generalized Hulthén potential has been obtained by Peng *et. al* [29]. Jia and co-workers have extended PT -symmetric quantum mechanics for the Dirac theory to the PDM formalism

[30, 31].

In this work, we intend to solve the effective-mass Dirac equation for the Woods-Saxon potential and investigate the scattering and bound state solutions. We also study the transmission resonances and give some results in the case of the low momentum limit.

The organization of this work is as follows. In Section II, we give the one-dimensional Dirac equation for the case of position-dependent mass and obtain a Schrödinger-like equation. In Section III, we calculate the transmission and reflection coefficients by analyzing the behavior of the wave functions at $x \rightarrow \mp\infty$. To compare our results, we also calculate the same coefficients for the case of constant mass and give our results. In Section IV, we study the bound state problem for the effective-mass Dirac equation and give the transmission resonance and bound state equations for the low momentum limit. Conclusions are given in Section V.

II. DIRAC EQUATION WITH POSITION DEPENDENT MASS

The relativistic free-particle Dirac equation ($\hbar = c = 1$) is written as [3, 15]

$$[i\gamma^\mu \partial_\mu - m(x)] \psi(x) = 0, \quad (1)$$

where we assume that the mass of the Dirac particle depends only on one spatially coordinate x . Under the effect of an external potential $V(x)$ and taking the gamma matrices γ_x and γ_0 as the Pauli matrices $i\sigma_x$ and σ_z , respectively, the Dirac equation in one-dimension becomes

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} - [E - V(x)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + m(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = 0, \quad (2)$$

which gives the two following couple differential equations

$$\frac{d\varphi_1(x)}{dx} = -[E - V(x) + m(x)] \varphi_2(x), \quad (3)$$

$$\frac{d\varphi_2(x)}{dx} = [E - V(x) - m(x)] \varphi_1(x). \quad (4)$$

The solutions can be more easily obtained by using a two-component approach introduced by Flügge as [32]

$$\phi(x) = \varphi_1(x) + i\varphi_2(x), \quad (5)$$

$$\chi(x) = \varphi_1(x) - i\varphi_2(x), \quad (6)$$

which leads

$$\frac{d\phi(x)}{dx} = i[E - V(x)]\phi(x) - im(x)\chi(x), \quad (7)$$

$$\frac{d\chi(x)}{dx} = -i[E - V(x)]\chi(x) + im(x)\phi(x). \quad (8)$$

Eliminating $\chi(x)$ in Eq. (7) and inserting into Eq. (8) and following the similar procedure for $\phi(x)$, we obtain two uncoupled second-order differential equations for $\phi(x)$ and $\chi(x)$, respectively

$$\begin{aligned} \frac{d^2\phi(x)}{dx^2} - \frac{dm(x)/dx}{m(x)} \frac{d\phi(x)}{dx} + \left\{ [E - V(x)]^2 - m^2(x) + i \frac{dV(x)}{dx} \right. \\ \left. + i[E - V(x)] \frac{dm(x)/dx}{m(x)} \right\} \phi(x) = 0, \end{aligned} \quad (9)$$

and

$$\begin{aligned} \frac{d^2\chi(x)}{dx^2} - \frac{dm(x)/dx}{m(x)} \frac{d\chi(x)}{dx} + \left\{ [E - V(x)]^2 - m^2(x) - i \frac{dV(x)}{dx} \right. \\ \left. - i[E - V(x)] \frac{dm(x)/dx}{m(x)} \right\} \chi(x) = 0. \end{aligned} \quad (10)$$

III. MASS FUNCTION AND SCATTERING STATE SOLUTIONS

We assume that the mass of the Dirac particle depends on spatially coordinate giving as

$$m(x) = m_0 + m_1 f(x), \quad (11)$$

where the function of x as $f(x) = 1/(1 + e^{\alpha(|x|-L)})$. The parameter m_0 will correspond to the rest mass of the particle and m_1 is a real, positive, small parameter. The mass form provides us to obtain the analytical results for the reflection and transmission coefficients and also the bound state solutions for the case of position-dependent mass and to analyze the results for the case of constant mass. On the other hand, from Eq. (11), it is easy to see that the ratio of the derivative of the mass to the mass is proportional with the mass parameter m_1 . So we ignore the terms that contain the derivative of the mass in Eqs. (9) and (10) for the case of $m_1 \rightarrow 0$ [29]. Under this assumption, Eqs. (9) and (10) become

$$\frac{d^2\phi(x)}{dx^2} + \left\{ [E - V(x)]^2 - m^2(x) + i \frac{dV(x)}{dx} \right\} \phi(x) = 0, \quad (12)$$

$$\frac{d^2\chi(x)}{dx^2} + \left\{ [E - V(x)]^2 - m^2(x) - i \frac{dV(x)}{dx} \right\} \chi(x) = 0. \quad (13)$$

We search the scattering states of the Dirac equation for the WS potential barrier [3]

$$V(x) = V_0 f(x), \quad (14)$$

where the function $f(x)$ is defined in Eq. (11) with V_0, α and L are real parameters. This potential is one of the most important potential models in quantum mechanics and has a main role, as an internuclear potential, in the coupled-channels calculations within the heavy-ion physics [33]. The nuclear optical-model potential including the Woods-Saxon potential is used to analyze the elastic scattering problem of nucleons and heavy particles [34]. It is worth to say that we deal with a potential form for $aL \gg 1$. In this case, the potential form now closely becomes a rectangular barrier.

A. Solutions for $x < 0$

Using Eqs. (11) and (14) and defining a new variable $y = (1 + e^{-\alpha(x+L)})^{-1}$, Eq. (12) turns into

$$y(1-y) \frac{d^2 \phi_L(y)}{dy^2} + (1-2y) \frac{d\phi_L(y)}{dy} + \frac{1}{y(1-y)} \{C_1 - C_2 y + C_3 y^2\} \phi_L(y) = 0, \quad (15)$$

where

$$C_1 = \frac{E^2 - m_0^2}{\alpha^2}; \quad C_2 = \frac{2V_0 E + 2m_0 m_1 - iV_0 \alpha}{\alpha^2}; \quad C_3 = \frac{V_0^2 - m_1^2 - iV_0 \alpha}{\alpha^2}. \quad (16)$$

In order to get a hypergeometric-type differential equation, we offer a trial function $\phi_L(y) = y^\mu (1-y)^\nu f(y)$. Substitution it into Eq. (15) leads to

$$y(1-y) \frac{d^2 f(y)}{dy^2} + [1 + 2\mu - 2(\mu + \nu + 1)y] \frac{df(y)}{dy} - (\mu + \nu + \sigma)(\mu + \nu - \sigma + 1)f(y) = 0, \quad (17)$$

with

$$\sigma = \frac{1}{2} + \sqrt{\left(\frac{iV_0}{\alpha} + \frac{1}{2}\right)^2 + \frac{m_1^2}{\alpha^2}}; \quad \nu = \frac{i}{\alpha} \sqrt{(E - V_0)^2 - (m_0 + m_1)^2}; \quad \mu = \frac{ik}{\alpha}; \quad k = \sqrt{E^2 - m_0^2}. \quad (18)$$

Eq. (17) has a general solution [35]

$$\begin{aligned} f(y) = & L_1 {}_2F_1(\mu + \nu + \sigma, \mu + \nu - \sigma + 1, 1 + 2\mu; y) \\ & + L_2 y^{-2\mu} {}_2F_1(-\mu + \nu + \sigma, -\mu + \nu - \sigma + 1, 1 - 2\mu; y), \end{aligned} \quad (19)$$

which gives

$$\phi_L(y) = L_1 y^\mu (1-y)^\nu {}_2F_1(a, b, c; y) + L_2 y^{-\mu} (1-y)^\nu {}_2F_1(a_1, b_1, c_1; y), \quad (20)$$

where

$$\begin{aligned} a &= \mu + \nu + \sigma ; & a_1 &= -\mu + \nu + \sigma , \\ b &= \mu + \nu - \sigma + 1 ; & b_1 &= -\mu + \nu - \sigma + 1 , \\ c &= 1 + 2\mu ; & c_1 &= 1 - 2\mu . \end{aligned} \quad (21)$$

B. Solutions for $x > 0$

In this case, changing the variable $z = 1/(1 + e^{\alpha(x-L)})$ and inserting Eqs. (11) and (14) into Eq. (12), we get

$$z(1-z) \frac{d^2 \phi_R(z)}{dz^2} + (1-2z) \frac{d\phi_R(z)}{dz} + \frac{1}{z(1-z)} \{C_4 - C_5 z + C_6 z^2\} \phi_R(z) = 0, \quad (22)$$

with

$$C_4 = \frac{E^2 - m_0^2}{\alpha^2} ; \quad C_5 = \frac{2E + 2m_0 m_1 + iV_0 \alpha}{\alpha^2} ; \quad C_6 = \frac{V_0^2 - m_1^2 + iV_0 \alpha}{\alpha^2}. \quad (23)$$

Defining a wave function of the form $\phi_R(z) = z^\tau (1-z)^\gamma h(z)$ in Eq. (22) gives a hypergeometric-type equation [35]

$$z(1-z) \frac{d^2 h(z)}{dz^2} + [1 + 2\tau - 2(\tau + \gamma + 1)z] \frac{dh(z)}{dz} - (\tau + \gamma + \delta)(\tau + \gamma - \delta + 1)h(z) = 0, \quad (24)$$

where we have obtained as $\tau = \mu, \gamma = \nu$ and $\delta = \frac{1}{2} + \sqrt{\left(\frac{iV_0}{\alpha} - \frac{1}{2}\right)^2 + \frac{m_1^2}{\alpha^2}}$. Eq. (24) has a solution in terms of hypergeometric functions [35]

$$\begin{aligned} h(z) &= R_1 {}_2F_1(\mu + \nu + \delta, \mu + \nu - \delta + 1, 1 + 2\mu; z) \\ &+ R_2 z^{-2\mu} {}_2F_1(-\mu + \nu + \delta, -\mu + \nu - \delta + 1, 1 - 2\mu; z), \end{aligned} \quad (25)$$

which gives the following complete solution for $x > 0$

$$\phi_R(z) = R_1 z^\mu (1-z)^\nu {}_2F_1(a_3, b_3, c_3; z) + R_2 z^{-\mu} (1-z)^\nu {}_2F_1(a_2, b_2, c_2; z), \quad (26)$$

where

$$\begin{aligned}
a_2 &= -\mu + \nu + \delta ; & a_3 &= \mu + \nu + \delta , \\
b_2 &= -\mu + \nu - \delta + 1 ; & b_3 &= \mu + \nu - \delta + 1 , \\
c_2 &= 1 - 2\mu ; & c_3 &= 1 + 2\mu .
\end{aligned} \tag{27}$$

If $x \rightarrow \infty$, $z \rightarrow 0$, then $(1 - z)^\nu \rightarrow 1$ and $z^\mu \rightarrow e^{-\alpha\mu(x-L)}$. Thus, we obtain the following right solution in this limit

$$\phi_R(x) \approx R_1 e^{-ik(x-L)} + R_2 e^{ik(x-L)}, \tag{28}$$

where we have used the following property of the hypergeometric functions: ${}_2F_1(\xi_1, \xi_2, \xi_3; t) \xrightarrow[t \rightarrow 0]{} 1$. In order to get a plane wave coming from the left to the right, we set $R_1 = 0$. Consequently, the right solution becomes

$$\phi_R(z) = R_2 z^{-\mu} (1 - z)^\nu {}_2F_1(a_2, b_2, c_2; z). \tag{29}$$

C. Reflection and Transmission Coefficients

Let us now study the behavior of the wave functions $\phi_L(y)$ and $\phi_R(z)$ at infinity to obtain the reflection and transmission coefficients. In the limit $x \rightarrow -\infty$, $y \rightarrow 0$, $(1 - y)^\nu \rightarrow 1$ and $y^\mu \rightarrow e^{\alpha\mu(x+L)}$ and in the limit $x \rightarrow \infty$, $z \rightarrow 0$, $(1 - z)^\nu \rightarrow 1$ and $z^\mu \rightarrow e^{-\alpha\mu(x-L)}$ as well (with ${}_2F_1(\xi_1, \xi_2, \xi_3; t) \xrightarrow[t \rightarrow 0]{} 1$), we have the wave functions, respectively,

$$\phi_L(x) \xrightarrow[x \rightarrow -\infty]{} L_1 e^{ik(x+L)} + L_2 e^{-ik(x+L)}, \tag{30}$$

$$\phi_R(x) \xrightarrow[x \rightarrow \infty]{} R_2 e^{ik(x-L)}. \tag{31}$$

In order to get the electrical current density for the one-dimensional Dirac equation defined by

$$j = \frac{1}{2} [|\phi(x)|^2 - |\chi(x)|^2], \tag{32}$$

we need to insert Eqs. (30) and (31) into Eq. (7) which gives $\chi_L(x)$ and $\chi_R(x)$, respectively,

$$\chi_L(x) = \frac{1}{m(x)} [(E - k)L_1 e^{ik(x+L)} + (E + k)L_2 e^{-ik(x+L)}], \tag{33}$$

$$\chi_R(x) = \left(\frac{E - k}{m(x)} \right) R_2 e^{ik(x-L)}. \tag{34}$$

The current in Eq. (32) can be written as $j_L = j_{in} - j_{refl}$ in the limit $x \rightarrow -\infty$ where j_{in} is the incident and j_{refl} is the reflected current. Similarly as $x \rightarrow \infty$ the current is $j_R = j_{trans}$ where j_{trans} is the transmitted current. Inserting Eqs. (30), (31), (33) and (34) into Eq. (32), we find the reflection and transmission coefficients, respectively, as

$$R = \frac{(E + k) |L_2|^2}{(E - k) |L_1|^2}, \quad (35)$$

$$T = \frac{|R_2|^2}{|L_1|^2}. \quad (36)$$

We can find more explicit expressions for the above coefficients by using the continuity conditio of the wave function at $x = 0$. For the limit $x \rightarrow 0$, we have $y \rightarrow 1$ and ${}_2F_1(\xi_1, \xi_2, \xi_3; t) \xrightarrow[t \rightarrow 0]{} 1$, the wave function $\phi_L(x)$ ($aL \gg 0$)

$$\phi_L(x) \xrightarrow[x \rightarrow 0]{} [L_1 S_1 e^{-\alpha\nu L} + L_2 S_2 e^{-\alpha\nu L}] e^{-\alpha\nu x} + [L_1 S_3 e^{\alpha\nu L} + L_2 S_4 e^{\alpha\nu L}] e^{\alpha\nu x}, \quad (37)$$

where we have used the following identity of the hypergeometric functions [35]

$$\begin{aligned} {}_2F_1(\xi_1, \xi_2, \xi_3; t) &= \frac{\Gamma(\xi_3)\Gamma(\xi_3 - \xi_2 - \xi_1)}{\Gamma(\xi_3 - \xi_1)\Gamma(\xi_3 - \xi_2)} {}_2F_1(\xi_1, \xi_2, \xi_1 + \xi_2 - \xi_3; 1 - t) \\ &+ (1 - t)^{\xi_3 - \xi_2 - \xi_1} \frac{\Gamma(\xi_3)\Gamma(\xi_1 + \xi_2 - \xi_3)}{\Gamma(\xi_1)\Gamma(\xi_2)} {}_2F_1(\xi_3 - \xi_1, \xi_3 - \xi_2, \xi_3 - \xi_2 - \xi_1 + 1; 1 - t). \end{aligned} \quad (38)$$

The abbreviations in Eq. (37) are

$$\begin{aligned} S_1 &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} ; \quad S_2 = \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)}, \\ S_3 &= \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} ; \quad S_4 = \frac{\Gamma(c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1)\Gamma(b_1)}. \end{aligned} \quad (39)$$

We have $z \rightarrow 1$ for the same limit $x \rightarrow 0$, so we write $\phi_R(x)$ as

$$\phi_R(x) \xrightarrow[x \rightarrow 0]{} R_2 [S_5 e^{\alpha\nu(x-L)} + S_6 e^{-\alpha\nu(x-L)}], \quad (40)$$

where

$$S_5 = \frac{\Gamma(c_2)\Gamma(c_2 - a_2 - b_2)}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)} ; \quad S_6 = \frac{\Gamma(c_2)\Gamma(a_2 + b_2 - c_2)}{\Gamma(a_2)\Gamma(b_2)}. \quad (41)$$

Finally, from matching the wave functions in Eqs. (37) and (40), we obtain

$$\frac{L_2}{L_1} = \frac{e^{-4\alpha\nu L} S_1 S_5 - S_3 S_6}{S_4 S_6 - e^{-4\alpha\nu L} S_2 S_5}, \quad (42)$$

and

$$\frac{R_2}{L_1} = \frac{e^{-2\alpha\nu L}[S_1S_4 - S_3S_2]}{S_4S_6 - e^{-4\alpha\nu L}S_2S_5}. \quad (43)$$

In Fig. 1, it is seen that the transmission and reflection coefficients oscillate between the values zero and one and satisfy the condition $R+T = 1$ for the case of constant and PDM as well. The oscillations appear in the range of $m_0 < E < 2m_0$ while $T = 0$ ($R = 1$) as $E < m_0$ and $T = 1$ ($R = 0$) as $E > 2m_0$. The effect of the PDM is just to shift the picks to the left. We present the variation of the transmission coefficient with respect to the parameter V_0 in the case of PDM in Fig. 2. We also plot the same variation for the constant mass. The transmission coefficient goes to zero with increasingly high of potential barrier and exactly zero for the values in the range $0.4 < V_0 < 1.2$. The coefficient T starts to oscillate and does not take zero-value for $V_0 > 1.2$ where the upper value of the oscillation for T is one. Fig. 2 shows also the effect of PDM on the dependence of T on potential parameter V_0 and that this effect is very weak. Fig. 3 shows the dependence of the transmission coefficient on the potential parameters α (left plot) and L (right plot) in the case of PDM in view of the varying particle energy. In both of the plots, T oscillates between the values zero and one as in the case of constant mass. It is seen that the frequency of the oscillations increases while the parameter L increases. In Fig. 4, we show the effects of the potential parameters α (left plot) and L (right plot) on the variation of the transmission coefficient with varying potential parameter V_0 . In these figures, coefficient of transmission is exactly zero within the range of $m_0 < V_0 < 3m_0$ and continue to oscillate out of this range.

One interesting point is the so-called "transmission resonances" appearing especially in relativistic domain [3, 6, 14, 15]. Within our present formalism, considering Eq. (42), the transmission resonances ($R = 0$, $T = 1$) occur when

$$e^{-4\alpha\nu L}S_1S_5 - S_3S_6 = 0. \quad (44)$$

From Figs. 1 and 2, we see that the Dirac particle has transmission resonances in the case of PDM like the case of constant mass. In both of figures, we observe that the effect of mass depending on coordinate is to shift the picks to the left. Fig. 3 shows that, as in the case of constant mass, transmission resonances appear. From the left plot of Fig. 3, one can observe that the width of the resonance peaks decreases and the number of the transmission resonances remains the same while the parameter α decreases. In addition, it is seen from

the left plot of the Fig. 3 that the first resonance peak appears at smaller values of the Dirac particle's energy in the presence of the small value of the parameter α . From the right panel of the Fig. 3, one observes that number of the resonance peaks decreases while the width of the resonance peaks increases with decreasing the parameter L . From Fig. 4, we see that transmission resonances could be observed with varying V_0 in the case of PDM and width of the resonance peaks decreases while the parameter α becomes smaller but it increases as the parameter L decreases.

IV. BOUND STATE SOLUTIONS

We tend to find the bound states for the Woods-Saxon potential well which means $V_0 \rightarrow -V_0$ in Eq. (14).

A. Solutions for $x < 0$

In order to get a complete solution for this region, we use a new variable $y = [1 + e^{-\alpha(x+L)}]^{-1}$ in Eq. (12) and take into account $V_0 \rightarrow -V_0$, we have

$$y(1-y)\frac{d^2\phi_L(y)}{dy^2} + (1-2y)\frac{d\phi_L(y)}{dy} + \frac{1}{y(1-y)}\{C'_1 - C'_2y + C'_3y^2\}\phi_L(y) = 0, \quad (45)$$

where

$$\underbrace{C'_1 = \frac{E^2 - m_0^2}{\alpha^2}}_{C'_1 = C_1} ; \underbrace{C'_2 = \frac{-2V_0E + 2m_0m_1 + iV_0\alpha}{\alpha^2}}_{C'_2 \xrightarrow{V_0 \rightarrow -V_0} C_2} ; \underbrace{C'_3 = \frac{V_0^2 - m_1^2 + iV_0\alpha}{\alpha^2}}_{C'_3 \xrightarrow{V_0 \rightarrow -V_0} C_3}. \quad (46)$$

Taking a trial wave function as $\phi_L(y) = y^{\mu'}(1-y)^{\nu'}g(y)$ and inserting it into Eq. (45) we obtain

$$y(1-y)\frac{d^2g(y)}{dy^2} + [1 + 2\mu' - 2(\mu' + \nu' + 1)y]\frac{dg(y)}{dy} - (\mu' + \nu' + \sigma')(\mu' + \nu' - \sigma' + 1)g(y) = 0, \quad (47)$$

with

$$\sigma' = \frac{1}{2} + \sqrt{\left(\frac{-iV_0}{\alpha} + \frac{1}{2}\right)^2 + \frac{m_1^2}{\alpha^2}} ; \nu' = \frac{i}{\alpha}\sqrt{(E + V_0)^2 - (m_0 + m_1)^2} ; \mu' = -\frac{1}{\alpha}\sqrt{m_0^2 - E^2}. \quad (48)$$

The solution of Eq. (47) is written in terms of the hypergeometric type functions [35]

$$\phi_L(y) = L_3 y^{\mu'} (1-y)^{\nu'} {}_2F_1(a', b', c'; y) + L_4 y^{-\mu'} (1-y)^{\nu'} {}_2F_1(a'_1, b'_1, c'_1; y) \quad (49)$$

where

$$\begin{aligned} a' &= \mu' + \nu' + \sigma' ; & a'_1 &= -\mu' + \nu' + \sigma' , \\ b' &= \mu' + \nu' - \sigma' + 1 ; & b'_1 &= -\mu' + \nu' - \sigma' + 1 , \\ c' &= 1 + 2\mu' ; & c'_1 &= 1 - 2\mu' . \end{aligned} \quad (50)$$

B. Solutions for $x > 0$

Inserting the potential function

$$V(x) = -\frac{V_0}{1 + e^{\alpha(x-L)}} , \quad (51)$$

into Eq. (12) and using the variable $z = 1/[1 + e^{\alpha(x-L)}]$, we get

$$z(1-z) \frac{d^2 \phi_R(z)}{dz^2} + (1-2z) \frac{d\phi_R(z)}{dz} + \frac{1}{z(1-z)} \{C'_4 - C'_5 z + C'_6 z^2\} \phi_R(z) = 0 , \quad (52)$$

with

$$\underbrace{C'_4 = \frac{E^2 - m_0^2}{\alpha^2}}_{C'_4 = C_4} ; \quad \underbrace{C'_5 = \frac{-2V_0 E + 2m_0 m_1 - iV_0 \alpha}{\alpha^2}}_{C'_5 \xrightarrow{V_0 \rightarrow -V_0} C_5} ; \quad \underbrace{C'_6 = \frac{V_0^2 - m_1^2 - iV_0 \alpha}{\alpha^2}}_{C'_6 \xrightarrow{V_0 \rightarrow -V_0} C_6} . \quad (53)$$

Taking a wave function of the form $\phi_R(z) = z^{\mu'} (1-z)^{\nu'} w(z)$ in Eq. (52) gives a hypergeometric-type equation [35]

$$z(1-z) \frac{d^2 w(z)}{dz^2} + [1 + 2\mu' - 2(\mu' + \nu' + 1)z] \frac{dw(z)}{dz} - (\mu' + \nu' + \delta')(\mu' + \nu' - \delta' + 1)w(z) = 0 , \quad (54)$$

where $\delta' = \frac{1}{2} + \sqrt{\left(\frac{-iV_0}{\alpha} - \frac{1}{2}\right)^2 + \frac{m_1^2}{\alpha^2}}$. The general solution of Eq. (54) is written in terms of hypergeometric functions as follow

$$\begin{aligned} w(z) &= R_3 {}_2F_1(\mu' + \nu' + \delta', \mu' + \nu' - \delta' + 1, 1 + 2\mu'; z) \\ &+ R_4 z^{-2\mu'} {}_2F_1(-\mu' + \nu' + \delta', -\mu' + \nu' - \delta' + 1, 1 - 2\mu'; z), \end{aligned} \quad (55)$$

and the whole solution for $x > 0$ is

$$\phi_R(z) = R_3 z^{\mu'} (1-z)^{\nu'} {}_2F_1(a'_3, b'_3, c'_3; z) + R_4 z^{-\mu'} (1-z)^{\nu'} {}_2F_1(a'_2, b'_2, c'_2; z) \quad (56)$$

where

$$\begin{aligned} a'_2 &= -\mu' + \nu' + \delta' ; & a'_3 &= \mu' + \nu' + \delta' , \\ b'_2 &= -\mu' + \nu' - \delta' + 1 ; & b'_3 &= \mu' + \nu' - \delta' + 1 , \\ c'_2 &= 1 - 2\mu' ; & c'_3 &= 1 + 2\mu' . \end{aligned} \quad (57)$$

Let us now extract the solutions given in Eqs. (49) and (56) in the limit $x \rightarrow \pm\infty$ to obtain the bound state wave function. Because of $y \rightarrow 0$ and $z \rightarrow 0$ as well for the limit $x \rightarrow \mp\infty$, we write ϕ_L in Eq. (49) and ϕ_R in Eq. (56), respectively

$$\phi_L(y) \xrightarrow{x \rightarrow -\infty} L_3 y^{\mu'} (1-y)^{\nu'} {}_2F_1(a', b', c'; y) , \quad (58)$$

$$\phi_R(z) \xrightarrow{x \rightarrow \infty} R_3 z^{\mu'} (1-z)^{\nu'} {}_2F_1(a'_3, b'_3, c'_3; z) \quad (59)$$

where we set $L_4 = R_4 = 0$ for obtaining the bound state eigenfunctions. In order to study the behavior of the solution at $x = 0$, we need the property of the hypergeometric functions given in Eq. (38). Recalling that for $x \rightarrow 0$, $y, z \rightarrow 1$ and $(1-y)^{\nu'} \approx e^{-\alpha\nu'(x+L)}$ while $(1-z)^{\nu'} \approx e^{\alpha\nu'(x-L)}$ and using Eq. (38), we write the wave functions

$$\phi_L(x) \xrightarrow{x \rightarrow 0} L_3 \left[S'_1 e^{-\alpha\nu'(x+L)} + S'_3 e^{\alpha\nu'(x+L)} \right] , \quad (60)$$

$$\phi_R(x) \xrightarrow{x \rightarrow 0} R_3 \left[S'_4 e^{\alpha\nu'(x-L)} + S'_2 e^{-\alpha\nu'(x-L)} \right] \quad (61)$$

where

$$\begin{aligned} S'_1 &= \frac{\Gamma(c')\Gamma(c' - a' - b')}{\Gamma(c' - a')\Gamma(c' - b')} ; S'_2 = \frac{\Gamma(c'_3)\Gamma(a'_3 + b'_3 - c'_3)}{\Gamma(a'_3)\Gamma(b'_3)} , \\ S'_3 &= \frac{\Gamma(c')\Gamma(a' + b' - c')}{\Gamma(a')\Gamma(b')} ; S'_4 = \frac{\Gamma(c'_3)\Gamma(c'_3 - a'_3 - b'_3)}{\Gamma(c'_3 - a'_3)\Gamma(c'_3 - b'_3)} . \end{aligned} \quad (62)$$

Matching the functions given in Eqs. (60) and (61) at $x = 0$ requiring the continuity of the wave function, comparing the coefficients of $e^{\pm\alpha\nu'x}$ and setting the coefficients determinant to zero, we obtain the following eigenvalue condition for the Woods-Saxon potential well

$$f[\alpha, V_0, m_0, m_1, E] = S'_2 S'_3 - S'_1 S'_4 e^{-4\alpha\nu' L} = 0 . \quad (63)$$

The above expression can be solved numerically and the energy eigenvalues E could be obtained by setting $\text{Re}[f[\alpha, V_0, m_0, m_1, E]] = 0$ and also $\text{Im}[f[\alpha, V_0, m_0, m_1, E]] = 0$ since $f[\alpha, V_0, m_0, m_1, E] = 0$ is complex. We search the numerical energy values for the interval $m - V_0 \leq E \leq m$, since we are interested in bound states. Figs. 5 and 6 show the real energy eigenvalues for the PDM and constant mass cases, respectively. The eigenvalues which are shown with arrows are the points on the E -axis where the $\text{Re}[f[\alpha, V_0, m_0, m_1, E]]$ -curve (solid line) and $\text{Im}[f[\alpha, V_0, m_0, m_1, E]]$ -curve (dotted line) cross. From the Figs. 5 and 6, one can see that the number of bound states in the case of PDM increases relative to the constant mass case.

Finally, let us study our results in the case of low momentum limit which has become an attractive topic especially in the relativistic domain [3, 6, 14, 15]. In this limit, the Dirac equation has two distinct states: One has the energy $E = m$ corresponding to particle state and the other one has the energy $E = -m$ corresponding to anti-particle state where m is the particle mass [3].

Firstly, we investigate the resonance equation for the values of $E = \pm m$. For this case, Eq. (44) becomes

$$e^{-4\alpha\nu L} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \frac{\Gamma(c_2)\Gamma(c_2-a_2-b_2)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(c_2)\Gamma(a_2+b_2-c_2)}{\Gamma(a_2)\Gamma(b_2)}, \quad (64)$$

where the arguments are obtained from Eqs. (21) and (27) as

$$\begin{aligned} a &= \nu(E \rightarrow \pm m) + \sigma ; & a_2 &= \nu(E \rightarrow \pm m) + \delta , \\ b &= \nu(E \rightarrow \pm m) - \sigma + 1 ; & b_2 &= \nu(E \rightarrow \pm m) - \delta + 1 , \\ c &= 1 ; & c_2 &= 1 . \end{aligned} \quad (65)$$

Secondly, we write the bound state equation given in Eq. (63) for $E = \pm m$ giving

$$e^{-4\alpha\nu' L} \frac{\Gamma(c')\Gamma(c'-a'-b')}{\Gamma(c'-a')\Gamma(c'-b')} \frac{\Gamma(c'_3)\Gamma(c'_3-a'_3-b'_3)}{\Gamma(c'_3-a'_3)\Gamma(c'_3-b'_3)} = \frac{\Gamma(c')\Gamma(a'+b'-c')}{\Gamma(a')\Gamma(b')} \frac{\Gamma(c'_3)\Gamma(a'_3+b'_3-c'_3)}{\Gamma(a'_3)\Gamma(b'_3)}, \quad (66)$$

where the arguments could be given from Eqs. (50) and (57) as

$$\begin{aligned} a' &= \nu'(E \rightarrow \pm m) + \sigma' ; & a'_3 &= \nu'(E \rightarrow \pm m) + \delta' , \\ b' &= \nu'(E \rightarrow \pm m) - \sigma' + 1 ; & b'_3 &= \nu'(E \rightarrow \pm m) - \delta' + 1 , \\ c' &= 1 ; & c'_3 &= 1 . \end{aligned} \quad (67)$$

Because of the substituting $V_0 \rightarrow -V_0$, we see that $\nu' = \nu$, $\sigma' = \sigma$ and $\delta' = \delta$, so

$$\begin{aligned} a' &= a ; & a'_3 &= a_2 , \\ b' &= b ; & b'_3 &= b_2 , \\ c' &= c ; & c'_3 &= c_2 . \end{aligned} \tag{68}$$

which means that the resonance and bound state equations are equal for the low momentum limit in both of the constant and position-dependent mass cases. This result supports the one in Ref. [3] which declares that the conditions for the tunnelling without reflection through a potential barrier $V(x)$ of a Dirac particle with small momentum (resonances) and supporting of the potential well $-V(x)$ a bound state energy $E = \pm m$ called supercriticality are the same. It should be noted that the transmission resonance of particles when they scatter off potential barriers is equivalent to the one of anti-particles when scattering off potential wells [6, 14, 15] and tunnelling of a Dirac particle through a potential barrier is strongly related to the Klein paradox [14, 15].

V. CONCLUSIONS

We have approximately solved the scattering and bound state problems in one-dimensional effective-mass Dirac equation for the Woods-Saxon potential and studied the problem by using an approximation in the mass distribution given as $m_1 \rightarrow 0$. Using this approximation, we have found reflection and transmission coefficients by analyzing the behavior of the functions $\phi_L(x)$ ($x < 0$) and $\phi_R(x)$ ($x > 0$) at $x \rightarrow \mp\infty$ within the framework of position-dependent mass formalism. It has been observed that the coefficients oscillate for the range where $E \gg m_0$ and $V_0 \gg m_0$ for both cases in which T and R change with energy and with potential parameter, respectively. The unitarity condition has also been checked numerically in the case of position-dependent mass. We have pointed out that the results for the case of constant mass are similar with the ones obtained in the literature [3]. Meanwhile, the energy eigenvalue equation has been found by using the wave function obtained by imposing the boundary condition of a bound state. We have also studied the effect of the mass varying with coordinate on transmission resonances and on the results which are obtained for the low momentum limit ($E = \pm m$). The relation between the transmission resonance and bound-state energy eigenvalues has been presented in the presence of

the position dependent mass case.

VI. ACKNOWLEDGMENTS

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- [1] A. O. Barut and W. Rasmussen, J. Phys. B:Atom. Molec. **6**, 1965 (1973).
 - [2] J. Y. Guo and X. Z. Fang, Can. J. Phys. **87**, 1021 (2009).
 - [3] P. Kennedy, J. Phys. A **35**, 689 (2002).
 - [4] F. Brau and C. Semay, Phys. Rev. E **59**, 1207 (1999).
 - [5] V. M. Villalba and C. Rojas, Phys. Lett. A **362**, 21 (2007).
 - [6] C. Rojas and V. M. Villalba, Phys. Rev. A **71**, 052101 (2005).
 - [7] G. Levai, P. Siegl and M. Znojil, J. Phys. A **42**, 295201 (2009).
 - [8] G. F. Wei, C. Y. Long and S. H. Dong, Phys. Lett. A **372**, 2592 (2008).
 - [9] A. D. Alhaidari, H. Bahlouli and M. S. Abdelmonem, Ann. Phys. **324**, 2561 (2009).
 - [10] L. D. Fadeev, Trudy Mat. Inst. Stekl. **73**, 314 (1964).
 - [11] P. Senn, Am. J. Phys. **56**, 916 (1988).
 - [12] M. S. Bianchi, J. Math. Phys. **35**, 2719 (1994).
 - [13] D. Bohm, *Quantum Mechanics* (Printice Hall, Englewood Cliffs, NJ), 1951.
 - [14] N. Dombey, P. Kennedy and A. Calogeracos, Phys. Rev. Lett. **85**, 1787 (2000).
 - [15] P. Kennedy and N. Dombey, J. Phys. A : Math. Gen. **35**, 6645 (2002).
 - [16] V. M. Villalba and C. Rojas, Int. J. Mod. Phys. A **21**, 313 (2006).
 - [17] S. H. Dong and M. Lozada-Cassou, Phys. Lett. A **330**, 168 (2004).
 - [18] V. M. Villalba and L. A. Gonzalez-Arraga, Phys. Scr. **81** 025010 (2010).
 - [19] G. Bastard, *Wave Mechanics Applied to Semiconductor Heterostructure* (Les Editions de Physique), 1998.
 - [20] L. Serra and E. Lipparini, Europhys. Lett. **40**, 667 (1997).
 - [21] F. A. de Saavedra, J. Boronat, A. Polls and A. Fabrocini, Phys. Rev. B **50**, 4248 (1994).
 - [22] A. R. Plastino, M. Casas and A. Plastino, Phys. Lett. A **281**, 297 (2001).
 - [23] A. D. Alhaidari, Int. J. Theor. Phys. **42**, 2999 (2003).
 - [24] A. D. Alhaidari, Phys. Lett. A **322**, 72 (2004).
 - [25] O. von Roos, Phys. Rev. B **27**, 7547 (1983).
 - [26] L. Dekar, L. Chetouani and T. F. Hammann, J. Math. Phys. **39**, 2551 (1998).
 - [27] A. D. Alhaidari, H. Bahlouli, A. Al-Hasan and M. S. Abdelmonem, Phys. Rev. A **75**, 062711 (2007).

- [28] O. Panella, S. Biondini and A. Arda, J. Phys. A **43**, 325302 (2010).
- [29] X. L. Peng, J. Y. Liu and C. S. Jia, Phys. Lett. A **352**, 478 (2006).
- [30] C. S. Jia and A. S. Dutra, Ann. Phys. **323**, 566 (2008).
- [31] C. S. Jia and A. S. Dutra, J. Phys. A:Math. and Gen. **39**, 11877 (2006).
- [32] S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag), 1974.
- [33] K. Hagino, M. Dasgupta, I. I. Gontchar, D. J. Hinda, C. R. Morton and J. O. Newton, Proceedings of the Fourth Italy-Japan Symposium on Heavy-Ion Physics, Tokyo, Japan (World Scientific, Singapore), pp. 87-98, 2002.
- [34] O. V. Bespalova, E. A. Romanovsky and T. I. Spasskaya, J. Phys. G **29**, 1193 (2003).
- [35] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications), 1965.

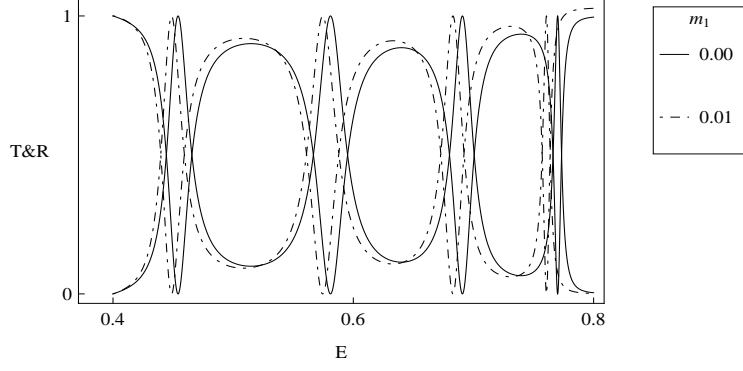


FIG. 1: transmission and reflection coefficients versus energy with $m_0 = 0.4, L = 10, \alpha = 5, V_0 = 1.2$.

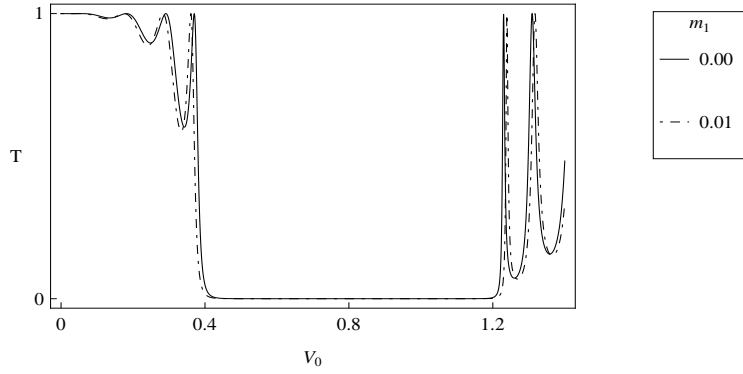


FIG. 2: coefficient of transmission versus potential parameter V_0 with $m_0 = 0.4, L = 10, \alpha = 5, E = 0.8$.

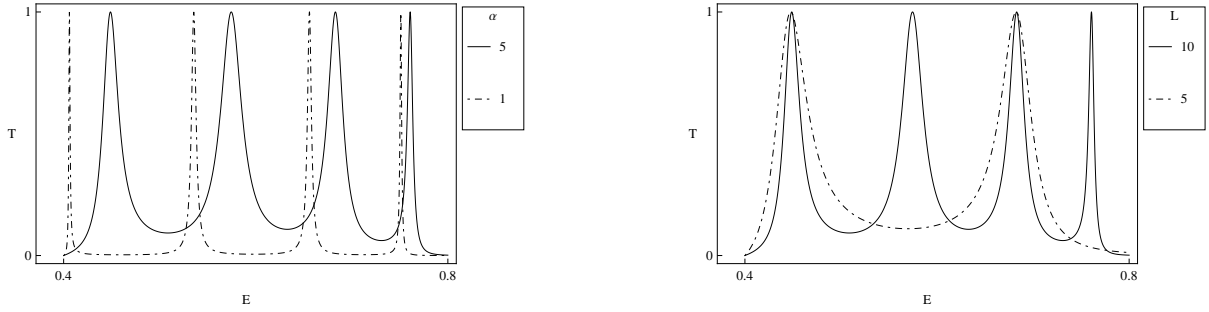


FIG. 3: transmission coefficient versus energy E with varying α (left plot: $m_0 = 0.4, L = 10, V_0 = 1.2$) and varying L (right plot: $m_0 = 0.4, \alpha = 5, V_0 = 1.2$) in case of PDM ($m_1 = 0.01$).

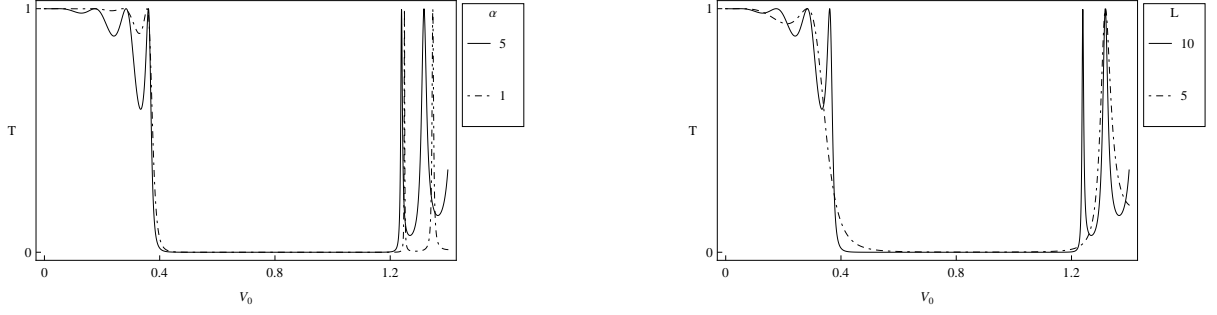


FIG. 4: transmission coefficient versus potential parameter V_0 with varying α (left plot: $m_0 = 0.4, L = 10, E = 0.8$) and varying L (right plot: $m_0 = 0.4, \alpha = 5, E = 0.8$) in case of PDM ($m_1 = 0.01$).

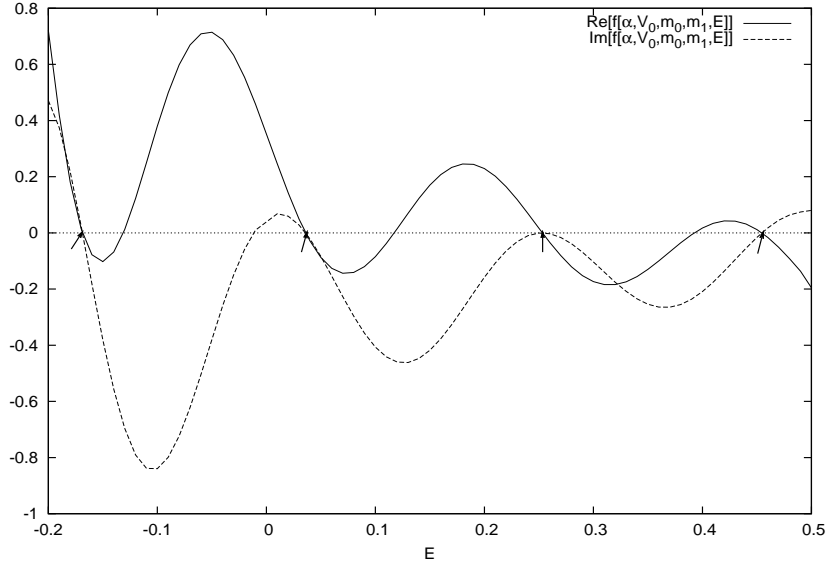


FIG. 5: plots of $\text{Re}[f[\alpha, V_0, m_0, m_1, E]]$ and $\text{Im}[f[\alpha, V_0, m_0, m_1, E]]$ versus E for $m_1 = 0.1$ ($m_0 = 0.5, L = 5, \alpha = 10, V_0 = 1$).

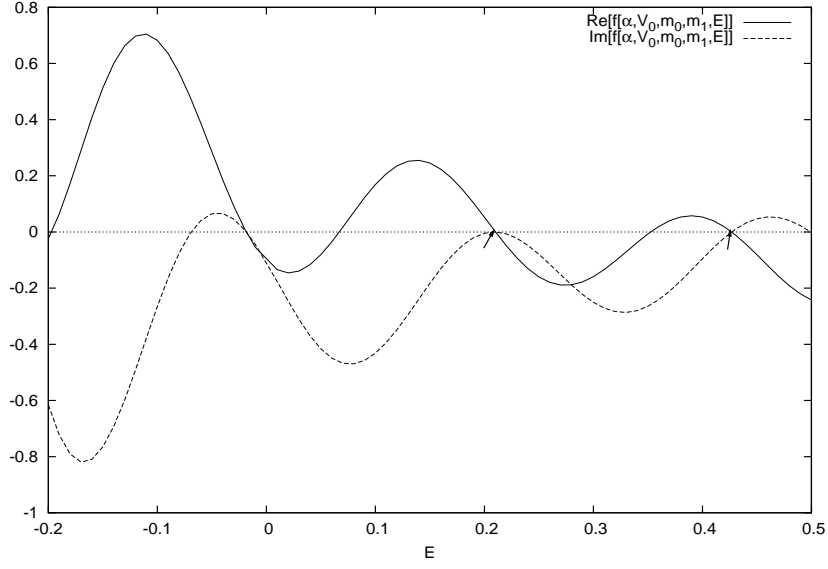


FIG. 6: plots of $\text{Re}[f[\alpha, V_0, m_0, m_1, E]]$ and $\text{Im}[f[\alpha, V_0, m_0, m_1, E]]$ versus E for the case of constant mass ($m_0 = 0.5, L = 5, \alpha = 10, V_0 = 1$).